ПЪРВА НАЦИОНАЛНА СТУДЕНТСКА ОЛИМПИАДА
ПО КОМПЮТЪРНА МАТЕМАТИКА
„АКАДЕМИК СТЕФАН ДОДУНЕКОВ"
ТЕХНИЧЕСКИ УНИВЕРСИТЕТ - ГАБРОВО
24-26. X. 2012 г.

Решения на задачите за група A

Задача 1

```
x = 5.1
y = 3.14
```
$\sqrt[3]{x+y} + \sqrt[3]{x-y}$
```
Remove[x, y]
```
5.1

3.14

3.27127

Задача 2

```
NSolve[x^2 - 5 * x + 11 == 0, x]
```
{{x → 2.5 − 2.17945 ⅈ}, {x → 2.5 + 2.17945 ⅈ}}

```
(2.5` - 2.179449471770337` ⅈ)^12 + (2.5` + 2.179449471770337` ⅈ)^12
```
$-2.41528 \times 10^{6} + 0.\ i$

```
Remove[x]
```

Задача 3

```
Factor[x^4 + 8 x^3 + 8 x - 1]
```
$(1 + x^2)\ (-1 + 8\ x + x^2)$

Solve$[-1 + 8\ x + x^2 == 0,\ x]$

$\{\{x \to -4 - \sqrt{17}\},\ \{x \to -4 + \sqrt{17}\}\}$

Следователно полиномът, разложен над R, има вида:
f(x) = (1 + x^2)(x + 4 + $\sqrt{17}$)(x + 4 - $\sqrt{17}$)

```
Remove[x]
```

Задача 4

```
PolynomialLCM[2 x^4 - 5 x^3 + x^2 - 10 x - 6, x^4 - 7 x^2 - 18]
```
$(1 + 2\ x)\ (-18 - 7\ x^2 + x^4)$

Collect[Expand[$(1 + 2\ x)\ (-18 - 7\ x^2 + x^4)$], x]

$-18 - 36\ x - 7\ x^2 - 14\ x^3 + x^4 + 2\ x^5$

```
Remove[x]
```

Задача 5

```
Reduce[(x + 2)^8 - (x - 2)^8 == 0, x, Complexes]
```

$x == 0 \;||\; x == -2\,i \;||\; x == 2\,i \;||\; x == -2\,i\sqrt{3-2\sqrt{2}} \;||$
$x == 2\,i\sqrt{3-2\sqrt{2}} \;||\; x == -2\,i\sqrt{3+2\sqrt{2}} \;||\; x == 2\,i\sqrt{3+2\sqrt{2}}$

```
Remove[x]
```

Задача 6

```
Factor[-x + x^9, Extension → {0, 1, 2}]
```

$(-1+x)\,x\,(1+x)\left(1+x^2\right)\left(1+x^4\right)$

```
Remove[x]
```

Задача 7

```
a = {{1, 1, x}, {0, 1, 1}, {0, 0, 1}}
b = {{1, 2012, x}, {0, 1, 2012}, {0, 0, 1}}
c = MatrixPower[a, 2012]
Solve[c == b, x]
```

{{1, 1, x}, {0, 1, 1}, {0, 0, 1}}

{{1, 2012, x}, {0, 1, 2012}, {0, 0, 1}}

{{1, 2012, 2 023 066 + 2012 x}, {0, 1, 2012}, {0, 0, 1}}

{{x → -1006}}

```
Remove[x, a, b, c]
```

Задача 8

```
a = {{1, 2, 3}, {2, 3, 1}, {3, 1, 2}}
b = {{2, 3, 4}, {3, 4, 2}, {4, 2, 3}}
c = {{3, 4, 5}, {4, 5, 3}, {5, 3, 4}}
x = Inverse[a].c.Inverse[b]
MatrixForm[x]
```

{{1, 2, 3}, {2, 3, 1}, {3, 1, 2}}

{{2, 3, 4}, {3, 4, 2}, {4, 2, 3}}

{{3, 4, 5}, {4, 5, 3}, {5, 3, 4}}

$\left\{\left\{-\frac{7}{27}, \frac{2}{27}, \frac{11}{27}\right\}, \left\{\frac{2}{27}, \frac{11}{27}, -\frac{7}{27}\right\}, \left\{\frac{11}{27}, -\frac{7}{27}, \frac{2}{27}\right\}\right\}$

$$\begin{pmatrix} -\frac{7}{27} & \frac{2}{27} & \frac{11}{27} \\ \frac{2}{27} & \frac{11}{27} & -\frac{7}{27} \\ \frac{11}{27} & -\frac{7}{27} & \frac{2}{27} \end{pmatrix}$$

```
Remove[x, a, b, c]
```

Задача 9

```
RSolve[{a[n] == a[n - 1] - 2 * a[n - 2], a[1] == 1, a[2] == 2 }, a[n], n]
```

$$\left\{\left\{a[n] \to -\left(-21\, i \left(\frac{1}{2} - \frac{i\sqrt{7}}{2}\right)^n + \sqrt{7}\left(\frac{1}{2} - \frac{i\sqrt{7}}{2}\right)^n + 14\, i \left(\frac{1}{2} + \frac{i\sqrt{7}}{2}\right)^n + 6\sqrt{7}\left(\frac{1}{2} + \frac{i\sqrt{7}}{2}\right)^n\right) \Big/ \left(14\left(-i + \sqrt{7}\right)\right)\right\}\right\}$$

$$\text{a102} = -\frac{1}{14\left(-i+\sqrt{7}\right)}\left(-21\, i \left(\frac{1}{2} - \frac{i\sqrt{7}}{2}\right)^n + \sqrt{7}\left(\frac{1}{2} - \frac{i\sqrt{7}}{2}\right)^n + 14\, i \left(\frac{1}{2} + \frac{i\sqrt{7}}{2}\right)^n + 6\sqrt{7}\left(\frac{1}{2} + \frac{i\sqrt{7}}{2}\right)^n\right) \ /.\ n \to 102$$

$$-\frac{1}{14\left(-i+\sqrt{7}\right)}\left(-21\, i \left(\frac{1}{2} - \frac{i\sqrt{7}}{2}\right)^{102} + \sqrt{7}\left(\frac{1}{2} - \frac{i\sqrt{7}}{2}\right)^{102} + 14\, i \left(\frac{1}{2} + \frac{i\sqrt{7}}{2}\right)^{102} + 6\sqrt{7}\left(\frac{1}{2} + \frac{i\sqrt{7}}{2}\right)^{102}\right)$$

$$\text{a101} = -\frac{1}{14\left(-i+\sqrt{7}\right)}\left(-21\, i \left(\frac{1}{2} - \frac{i\sqrt{7}}{2}\right)^n + \sqrt{7}\left(\frac{1}{2} - \frac{i\sqrt{7}}{2}\right)^n + 14\, i \left(\frac{1}{2} + \frac{i\sqrt{7}}{2}\right)^n + 6\sqrt{7}\left(\frac{1}{2} + \frac{i\sqrt{7}}{2}\right)^n\right) \ /.\ n \to 101$$

$$-\frac{1}{14\left(-i+\sqrt{7}\right)}\left(-21\, i \left(\frac{1}{2} - \frac{i\sqrt{7}}{2}\right)^{101} + \sqrt{7}\left(\frac{1}{2} - \frac{i\sqrt{7}}{2}\right)^{101} + 14\, i \left(\frac{1}{2} + \frac{i\sqrt{7}}{2}\right)^{101} + 6\sqrt{7}\left(\frac{1}{2} + \frac{i\sqrt{7}}{2}\right)^{101}\right)$$

```
r = MatrixPower[{{1, -2}, {1, 0}}, 100].{2, 1}
{-832 162 467 610 076, 1 286 059 417 395 116}

Reduce[{a102, a101} == r]
True

Remove[a, n, r, a101, a102]
```

Задача 10

```
Solve[{λ * x1 + x2 + x3 + x4 == 1, x1 + λ * x2 + x3 + x4 == 1,
  x1 + x2 + λ * x3 + x4 == 1, x1 + x2 + x3 + λ * x4 == 1}, {x1, x2, x3, x4}]
```

$$\left\{\left\{x1 \to -\frac{1}{-3-\lambda},\ x2 \to \frac{1}{3+\lambda},\ x3 \to \frac{1}{3+\lambda},\ x4 \to \frac{1}{3+\lambda}\right\}\right\}$$

```
Remove[x1, x2, x3, x4, λ]
```

Задача 11

Намираме координатите на връх А:

```
Reduce[x + 11 y - 23 == 0 && 10 x - y - 8 == 0, {x, y}]
```

x == 1 && y == 2

```
a = {1, 2}
```

{1, 2}

Намираме координатите на връх B:

```
Reduce[x + 11 y - 23 == 0 && 11 x + 10 y - 142 == 0, {x, y}]
```

x == 12 && y == 1

```
b = {12, 1}
```

{12, 1}

Намираме координатите на връх C:

```
Reduce[10 x - y - 8 == 0 && 11 x + 10 y - 142 == 0, {x, y}]
```

x == 2 && y == 12

Символът C е защитен; използваме C1 за връх C:

```
C1 = {2, 12}
```

{2, 12}

```
ab = b - a
ac = C1 - a
```

{11, -1}

{1, 10}

```
s = 0.5 Abs[Det[{ab, ac}]]
```

55.5

```
Remove[x, y, a, b, C1, A, B, ab, ac, AB, AC]
```

Задача 12

```
Solve[{(x - 4) ^2 + (y - 3) ^2 == 10 , (x - 7) ^2 + (y - 5) ^2 == 9}, {x, y}]
```

$\left\{\left\{x \to \frac{55}{13}, y \to \frac{80}{13}\right\}, \{x \to 7, y \to 2\}\right\}$

```
sreda = 0.5 * ({55 / 13, 80 / 13} + {7, 2})
```

{5.61538, 4.07692}

```
Remove[x, y, sreda]
```

Задача 13

```
pZY = {0, 3, -2}
pXZ = {1, 0, -2}
pXY = {1, 3, 0}
n = {x, y, z}
Solve[Det[{n - pZY, n - pXZ, n - pXY}] == 0, {x, y, z}]
```

{0, 3, -2}

{1, 0, -2}

{1, 3, 0}

{x, y, z}

Solve::svars : Equations may not give solutions for all "solve" variables. ≫

$$\left\{\left\{z \to -4 + 2\,x + \frac{2\,y}{3}\right\}\right\}$$

Следователно уравнението на равнината има вида:
2x + (2/3)y - z - 4 = 0

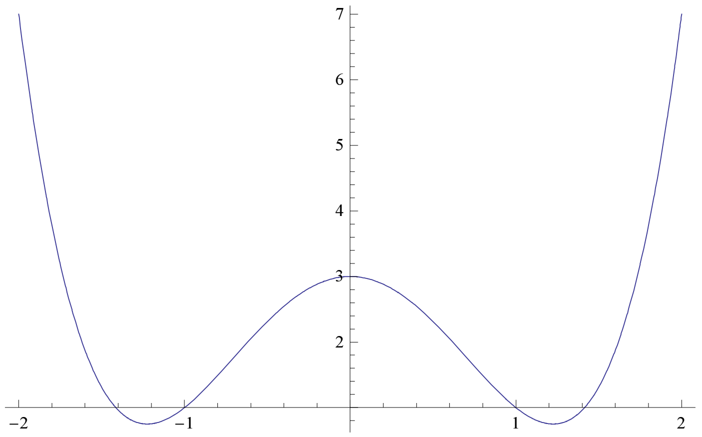
```
Remove[pZY, pXZ, pXY, x, y, z, n]
```

Задача 14

```
Solve[a * 16 - (a + 2) * 4 + 3 == 7, a]
```

{{a → 1}}

```
Plot[x^4 - 3 * x^2 + 3, {x, -2, 2}]
```




```
Remove[a, x]
```

Задача 15

```
Limit[(1 + Cot[x]) ^ (Tan[x]), x → π / 2]
```

e

```
Remove[x]
```

Задача 16

```
f[x_] := Log[Sin[2 * x] + Cos[3 * x]]
f''[π / 6]
```

$$-\frac{28}{3}$$

```
Remove[f, x]
```

Задача 17

```
f[x_] := Det[
  {{1, x, x, x, x}, {x, 2, x, x, x}, {x, x, 3, x, x}, {x, x, x, 4, x}, {x, x, x, x, 5}}]
FindMaximum[{f[p], 1 ≤ p ≤ 5}, p]
FindMinimum[{f[q], 1 ≤ q ≤ 5}, q]
```

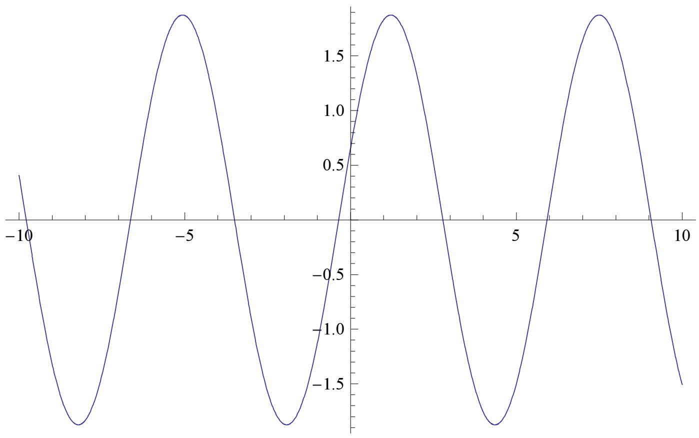
```
{24., {p → 1.}}
```

```
{-29.0985, {q → 4.22474}}
```

```
Remove[f, x, p, q]
```

Задача 18

```
Plot[Sin[x] - Cos[4 - x], {x, -10, 10}]
```



Sin[x]-Cos[x]=0 има безброй много решения, а най-малкото положително се намира в интервала [0,5]

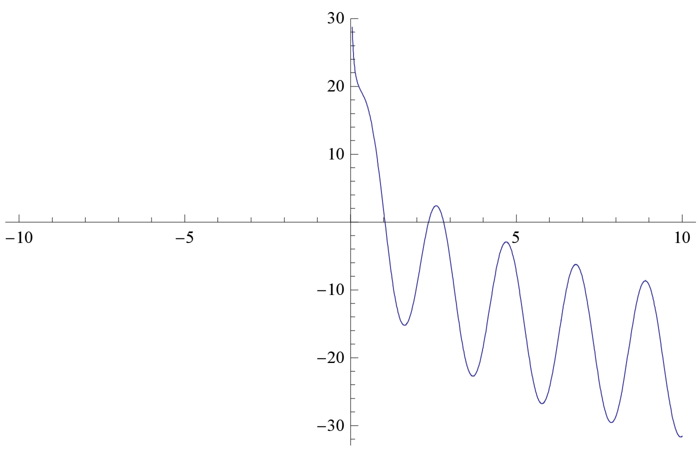
```
NSolve[Sin[x] - Cos[4 - x] == 0 && x ≥ 0 && x ≤ 5, x]
```

```
{{x → 2.7854}}
```

```
Remove[x]
```

Задача 19

```
Plot[11 * Sin[3 * x] - 9 * Log[x], {x, -10, 10}]
```



От графиката се вижда, че уравнението има 3 корена, близо до точките 1, 2.5 и 3

```
FindRoot[11 * Sin[3 * x] == 9 * Log[x], {x, 1}]
FindRoot[11 * Sin[3 * x] == 9 * Log[x], {x, 2.5}]
FindRoot[11 * Sin[3 * x] == 9 * Log[x], {x, 3}]
```

{x → 1.03723}

{x → 2.3529}

{x → 2.8065}

```
Remove[x]
```

Задача 20

```
z = D[Sin[x] + 3 * a * Sin[2 x] - 1 / 3 Sin[3 x] - 6 a * x, x]
```

-6 a + Cos[x] + 6 a Cos[2 x] - Cos[3 x]

```
TrigFactor[z]
```

$16\,\mathrm{Cos}\left[\frac{x}{2}\right]^2 (-3a + \mathrm{Cos}[x])\,\mathrm{Sin}\left[\frac{x}{2}\right]^2$

```
Reduce[-3 a + Cos[x] ≥ 0, x, Reals]
```

$a \le -\frac{1}{3}\ ||$

$\left(C[1] \in \mathrm{Integers}\ \&\&\ -\frac{1}{3} < a < \frac{1}{3}\ \&\&\ -\mathrm{ArcCos}[3a] + 2\pi C[1] \le x \le \mathrm{ArcCos}[3a] + 2\pi C[1]\right)\ ||$

$\left(C[1] \in \mathrm{Integers}\ \&\&\ a == \frac{1}{3}\ \&\&\ x == 2\pi C[1]\right)$

```
Remove[z, x, a]
```

Задача 21

Разглеждаме функцията в интервала x>=1 и интегрираме

```
f1 = Integrate[x - 1, x]
```

$-x + \frac{x^2}{2}$

Разглеждаме функцията в интервала (-∞;1] и интегрираме

```
f2 = Integrate[1 - x, x]
```

$x - \frac{x^2}{2}$

Тогава примитивната функция има вида:

```
F = Piecewise[{{f1, x ≥ 1}, {f2, x ≤ 1}}]
```

$$\begin{cases} -x + \frac{x^2}{2} & x \ge 1 \\ x - \frac{x^2}{2} & x \le 1 \\ 0 & \mathrm{True} \end{cases}$$

```
Remove[x, f1, F]
```

Задача 22

```
Integrate[1 / (1 + Sin[x] + Cos[x]), {x, 0, Pi / 2}]
```

Log[2]

```
Remove[x]
```

Задача 23

$$f[t\_] := Det\left[Table\left[\int_{\pi/4}^{t} \frac{Cos[x] + Sin[x]}{i + j}\, dx,\ \{i, 3\},\ \{j, 3\}\right]\right]$$

```
Series[f[s], {s, 0, 10}]
```

$$-\frac{1}{43200} + \frac{s}{14400} - \frac{s^2}{28800} - \frac{s^3}{17280} + \frac{13 s^4}{345600} + \frac{41 s^5}{1728000} - \frac{121 s^6}{10368000} - \frac{73 s^7}{14515200} + \frac{1093 s^8}{580608000} + \frac{3281 s^9}{5225472000} - \frac{9841 s^{10}}{52254720000} + O[s]^{11}$$

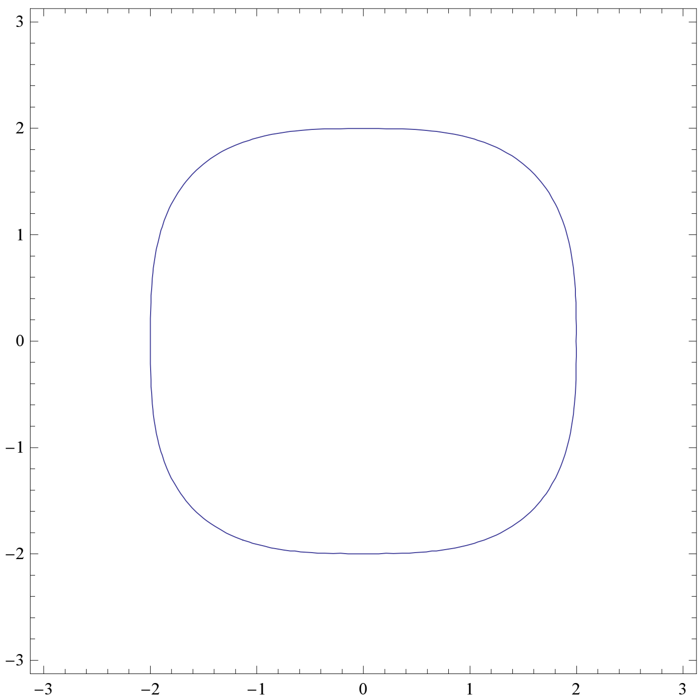
```
Normal[%]
```

$$-\frac{1}{43200} + \frac{s}{14400} - \frac{s^2}{28800} - \frac{s^3}{17280} + \frac{13 s^4}{345600} + \frac{41 s^5}{1728000} - \frac{121 s^6}{10368000} - \frac{73 s^7}{14515200} + \frac{1093 s^8}{580608000} + \frac{3281 s^9}{5225472000} - \frac{9841 s^{10}}{52254720000}$$

```
Remove[x, s, t, i, j, f]
```

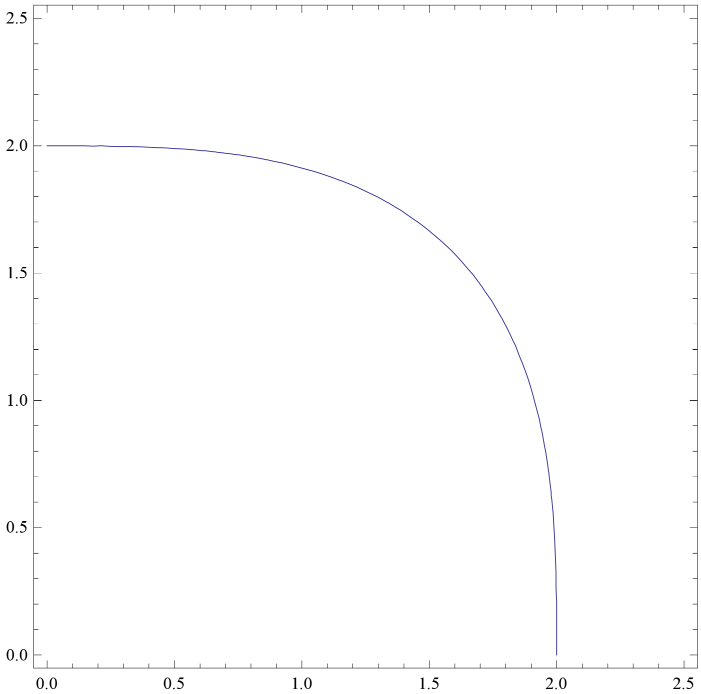
Задача 24

```
ContourPlot[Abs[x]^3 + Abs[y]^3 == 8, {x, -3, 3}, {y, -3, 3}]
```



Лицето на цялата област е 4 пъти лицето на областта под тази крива:

```
ContourPlot[x^3 + y^3 == 8, {x, 0, 2.5}, {y, 0, 2.5}]
```




```
S = 4 * Integrate[Integrate[x^3 + y^3, {x, 0, 2}], {y, 0, 2}]
64

Remove[x, y, S, s1]
```

Задача 25

$$\text{Solve}\left[\sum_{k=1}^{n} \frac{1}{k*(k+1)\,(k+2)} == \frac{1013545}{4054182} \;\&\&\; n \geq 0,\; n,\; \text{Integers}\right]$$

```
{{n → 2012}}

Remove[n, k]
```

Задача 26

$$\text{Solve}\left[\text{Limit}\left[\sum_{k=1}^{n} \text{Cos}[k*x],\; x \to \pi/3\right] == 1/2 \;\&\&\; n \geq 2000 \;\&\&\; n \leq 2012,\; n\right]$$

```
{{n → 2005}, {n → 2005}, {n → 2011}, {n → 2011}}

Remove[n, k, x]
```

Задача 27

```
Reduce[Integrate[(a * t + b * t^2) Cos[m * t], {t, 0, Pi}] == 1 / (m^2) &&
  m > 0 && m ∈ Integers, {a, b}, Reals]
```

$$m \in \text{Integers} \;\&\&\; m \geq 1 \;\&\&\; b == \frac{m + a\,m - a\,m\,\text{Cos}[m\,\pi] - a\,m^2\,\pi\,\text{Sin}[m\,\pi]}{2\,m\,\pi\,\text{Cos}[m\,\pi] - 2\,\text{Sin}[m\,\pi] + m^2\,\pi^2\,\text{Sin}[m\,\pi]}$$

Следователно, за $b == \frac{m + a\,m - a\,m\,\text{Cos}[m\,\pi] - a\,m^2\,\pi\,\text{Sin}[m\,\pi]}{2\,m\,\pi\,\text{Cos}[m\,\pi] - 2\,\text{Sin}[m\,\pi] + m^2\,\pi^2\,\text{Sin}[m\,\pi]}$ равенството е изпълнено за всяко m цяло положително.

```
Remove[a, b, m, t]
```

Задача 28

```
y[x_] := Cos[x] / 2 + (e^x + e^(-x)) / 4
```

```
Integrate[(x - t) y[t], {t, 0, x}]
```

$$\frac{1}{2} (-\text{Cos}[x] + \text{Cosh}[x])$$

$$\text{Reduce}\left[\text{Cos}[x] / 2 + (e\verb|^|x + e\verb|^|(-x)) / 4 == \text{Cos}[x] + \frac{1}{2} (-\text{Cos}[x] + \text{Cosh}[x]), x\right]$$

```
True
```

Следователно равенството е изпълнено за всяко x -> y(x) е решение на уравнението.

Задача 29

Търсим последните три цифри на числото 7^(7^2011), което е прекалено голямо, за да бъде изчислено директно. Търсим периода на 7^k по модул 1000.

```
Table[Mod[7^k, 1000], {k, 50}]
```

```
{7, 49, 343, 401, 807, 649, 543, 801, 607, 249, 743, 201, 407, 849, 943, 601, 207,
 449, 143, 1, 7, 49, 343, 401, 807, 649, 543, 801, 607, 249, 743, 201, 407, 849,
 943, 601, 207, 449, 143, 1, 7, 49, 343, 401, 807, 649, 543, 801, 607, 249}
```

Виждаме, че периодът p на 7^k по модул 1000 е 20, тоест mod(7^k, 1000) == mod(7^(k+20), 1000). Търсим остатъка при делението на 7^2011(степенният показател в основния израз) с периода p.

```
Mod[7^2011, 20]
```

```
3
```

Следователно mod(7^(7^2011), 1000) == mod(7^3, 1000)

```
Mod[7^3, 1000]
```

```
343
```

Последните три цифри на 7^(7^2011) са 3,4 и 3.

```
Remove[k]
```

Задача 30

```
Reduce[Mod[a^2, 10000] == 2016 && a ≥ 1000 && a ≤ 9999 && a ∈ Integers, a]
```

```
a == 1504 || a == 3496 || a == 4004 || a == 5996 || a == 6504 || a == 8496 || a == 9004
```

```
Remove[a]
```